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THE DESCRIPTION OF DENDRIFORM ALGEBRA STRUCTURES ON TWO-DIMENSIONAL COMPLEX SPACE

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Abstract

In this paper, we classify all dendriform algebra structures on two-dimensional complex space. We distinguish twelve isomorphism classes (one parametric family and eleven concrete) of two-dimensional complex dendriform algebras, and show that they exhaust all possible cases.

1. Introduction

Recall that a dendriform algebra is an associative algebra with two non-associative operations, written $\prec$ and $\succ$ satisfying three rules. The two products add to form the product of the algebra. At the same time, they define a left and right pre-Lie product on the same algebra. Loday has introduced this notion in connection with dialgebra structures [5].

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Besides of Loday's motivations, the key point from our perspective is the intimate relation between the so-called Rota-Baxter algebras and such dendriform algebras.

A Rota-Baxter algebra is an algebra A over a field K with a linear endomorphism β satisfying the Rota-Baxter relation:

$$\beta(x)\beta(y) = \beta(\beta(x)y + x\beta(y)) + \lambda\beta(xy), \quad \forall x,\ y \in A, \tag{1.1}$$

where λ is a fixed element of K. The map β is called a *Rota-Baxter operator* of weight λ. Associative Rota-Baxter algebras arise in many mathematical contexts, e.g., in integral and finite differences calculus, but also in perturbative renormalization in quantum field theory. The Rota-Baxter algebra was introduced by Baxter [3] in his probability study, and was popularized mainly by the investigations of Rota [8] and his colleagues.

It is known [6] that Rota-Baxter operators are closely related to dendriform algebras. Let E be an algebra equipped with two binary operations $\prec$ and $\succ$. Then $(E, \prec, \succ)$ is said to be a *dendriform algebra*, if

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$x \succ (y \succ z) = (x \prec y + x \succ y) \succ z,$$

$\forall x,\ y,\ z \in E.$

In [6], Loday has noted that the operad of dendriform algebras is the Koszul dual of the one of associative dialgebras. The dendriform algebra characterizes an associative multiplication, i.e., the sum of two multiplications, $x * y := x \prec y + x \succ y$, is associative. According to [1], one may associate a dendriform algebra to any associative algebra equipped with a Rota-Baxter operator, i.e., it was shown that, if A is associative algebra, then the pair of multiplications $x \prec y := \beta(x)y$ and $x \succ y := x\beta(y)$ give a dendriform structure on A, where $\beta : A \to A$ is a Rota-Baxter operator [3].

Dendriform algebras were introduced by Loday [6] with motivation from algebraic K-theory, and have been further studied with connections to several areas in mathematics and physics, including operads, homology, Hopf algebras, Lie and Leibniz algebras, combinatorics, arithmetic, and quantum field theory. In [6], Loday showed relationships of the dendriform algebras with other classes, such as Zinbiel, associative, diassociative, Lie and Leibniz algebras.

The classification of any class of algebras is a fundamental and very difficult problem. It is one of the first problems that one encounters when trying to understand the structure of this class of algebras. In this paper, we are interested in classification of complex dendriform algebras in low dimension.

Organization of this paper is as follows. Section 1 is a brief introduction. In Section 2, we define several kinds of nilpotency for dendriform algebras, and then prove that they all are equivalent. Unlike the diassociative algebras case [2], for a dendriform algebra to be nilpotent, it is not enough to be nilpotent with respect to one of the binary operations in it. The main result of the paper on classification of two-dimensional complex dendriform algebras is contained in Section 3.

Through the paper, all algebras assumed to be over the field of complex numbers $\mathbb{C}$.

All the results of this section have appeared elsewhere, particularly in [6].

Definition 1.1. Zinbiel algebra R is an algebra with a binary operation $\cdot : R \times R \to R,$ satisfying the condition:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y), \quad \text{for } \forall x,\ y,\ z \in R.$$

Loday has showed that the classical relationship between Lie and associative algebras can be translated into an analogous relationship between Zinbiel and associative-commutative algebras, with bracket $[x,\ y] = xy + yx.$

Definition 1.2. Dendriform algebra E is an algebra equipped with two binary operations

$$\succ: E \times E \to E, \quad \prec: E \times E \to E,$$

which $\forall x,\ y,\ z \in E$ satisfy the following axioms:

$$(x \prec y) \prec z = (x \prec y) \prec z + x \prec (y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z + (x \succ y) \succ z = x \succ (y \succ z).$$

Lemma 1.1. *For a dendriform algebra E, the product defined by*

$$x * y = x \prec y + x \succ y,$$

is associative.

Lemma 1.2. *Let R be a Zinbiel algebra and put*

$$x \prec y := x \cdot y, \quad x \succ y := y \cdot x, \quad \forall x,\ y \in R.$$

Then $(R, \prec, \succ)$ *is a dendriform algebra. Conversely, a commutative dendriform algebra E (i.e., a dendriform algebra for which* $xy = yx$ *) is a Zinbiel algebra.*

There is a commutative diagram of categories and functors given by Loday [6] as shown in Figure 1, where *Zinb*, *Com*, *As*, *Lie*, *Leib* stand for Zinbiel, Commutative-Associative, Associative, Lie and Leibniz algebras categories, respectively.

Let A be an n-dimensional associative algebra and $e_1, e_2, \dots, e_n$ be a basis of A. An algebra structure on A is determined by a set of structure constants γ_{ij}^k, where

$$e_i e_j = \sum\nolimits_k \gamma_{ij}^k e_k, \text{ for } 1 \le i,\ j,\ k \le n. \tag{1.2}$$

An algebra A is associative, if and only if the structure constants of it satisfy the following identities:

$$\gamma_{ij}^s \gamma_{sk}^t = \gamma_{jk}^s \gamma_{is}^t, \text{ where } 1 \le i,\ j,\ k,\ t \le n.$$

In Section 3, we shall use the following theorem from [2] on classification of 2-dimensional complex associative algebras.

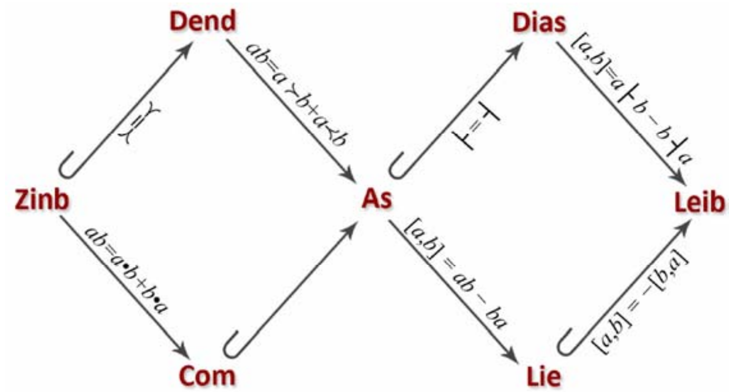

Figure 1. Diagram.

Theorem 1.1. *Any* 2*-dimensional complex associative algebra can be included in one of the following isomorphism classes of algebras*:

$$As_2^1 : e_1e_1 = e_2;$$

$$As_2^2 : e_1e_1 = e_1,\ e_1e_2 = e_2;$$

$$As_2^3 : e_1e_1 = e_1,\ e_2e_1 = e_2;$$

$$As_2^4 : e_1e_1 = e_1,\ e_2e_2 = e_2;$$

$$As_2^5 : e_1e_1 = e_1,\ e_1e_2 = e_2,\ e_2e_1 = e_2,$$

As_n^q *- stands for q-th associative algebra structure in n-dimensional vector space*.

2. More on Dendriform Algebras

Let E be an n-dimensional dendriform algebra and $e_1, e_2, \dots, e_n$ be a basis of E. An algebra structure on E is determined by a set of structure constants α_{ij}^k and β_{lp}^q, where

$$e_i \prec e_j = \sum_s \alpha_{ij}^s e_s, \quad e_l \succ e_p = \sum_t \beta_{lp}^t e_t, \text{ for } 1 \le i,\ j,\ l,\ p \le n. \quad (2.1)$$

The dendriform algebra axioms (Definition 2.1) give the following constraints for the structure constants α_{ij}^{k} and β_{lp}^{q}:

$$\alpha_{ij}^{s}\alpha_{sk}^{t} = \alpha_{jk}^{s}\alpha_{is}^{t} + \alpha_{is}^{t}\beta_{jk}^{s},$$

$$\alpha_{sk}^{t}\beta_{ij}^{s} = \alpha_{jk}^{s}\beta_{is}^{t},$$

$$\alpha_{ij}^{s}\beta_{sk}^{t} + \beta_{ij}^{s}\beta_{sk}^{t} = \beta_{is}^{t}\beta_{jk}^{s},$$

$1 \leq i, j, k, s, t \leq n.$

Lemma 2.1. *Let* $(E, \prec, \succ)$ *be a dendriform algebra. If one of the binary operations* $\prec, \succ$ *in E vanishes, then with respect to the second one, the algebra E is associative.*

Proof. The proof of this is straightforward.

Let E be a dendriform algebra. Let M and N be subsets of E. We define

$$M \lozenge N := M \prec N + M \succ N,$$

where $M \prec N = \{\sum \alpha_{ij} d_i \prec d_j | \alpha_{ij} \in \mathbb{C}, d_i \in M, d_j \in N\}$,

$$M \succ N = \{\sum \beta_{ij} d_k \succ d_s | \beta_{ks} \in \mathbb{C}, d_k \in M, d_s \in N\}.$$

Let us consider the following series:

$$E^{<1>} = E, E^{<k+1>} = E^{<k>} \lozenge E, \tag{2.2}$$

$$E^{\{1\}} = E, E^{\{k+1\}} = E \lozenge E^{\{k\}}, \tag{2.3}$$

$$E^{1} = E, E^{k+1} = E^{1} \lozenge E^{k} + E^{2} \lozenge E^{k-1} + \ldots + E^{k} \lozenge E^{1}. \tag{2.4}$$

It is clear that

$$E^{<1>} \supseteq E^{<2>} \supseteq E^{<3>} \supseteq \ldots,$$

$$E^{\{1\}} \supseteq E^{\{2\}} \supseteq E^{\{3\}} \supseteq \ldots,$$

and

$$E^1 \supseteq E^2 \supseteq E^3 \supseteq \dots .$$

Definition 2.1. Dendriform algebra E is said to be *right nilpotent* (respectively, left nilpotent), if there exists $k \in \mathbb{N} \,(p \in \mathbb{N})$ such that $E^{<k>} = 0 \,(E^{\{p\}} = 0)$.

Definition 2.2. Dendriform algebra E is said to be *nilpotent*, if there exists $s \in \mathbb{N}$ such that $E^s = 0$.

Lemma 2.2. *For any* $g,\ h \in \mathbb{N}$, *the following inclusions hold*:

(a)
$$E^{<g>} \lozenge E^{<h>} \subseteq E^{<g+h>}, \tag{2.5}$$

(b)
$$E^{\{g\}} \lozenge E^{\{h\}} \subseteq E^{\{g+h\}}. \tag{2.6}$$

Proof. The proof is carried out by induction on h for arbitrary g. The validity of (a) for $h = 1$ is trivial. Suppose that, it is true for h and prove it for $h + 1$. Then the induction argument, dendriform algebra axioms, and the following chain of equalities give the result:

$$\begin{aligned}
E^{<g>} \lozenge E^{<h+1>} &= E^{<g>} \lozenge (E^{<h>} \lozenge E^{<1>}) = E^{<g>} \lozenge (E^{<h>} \prec E^{<1>} + E^{<h>} \succ E^{<1>}) \\
&= E^{<g>} \prec (E^{<h>} \prec E^{<1>} + E^{<h>} \succ E^{<1>}) + E^{<g>} \succ (E^{<h>} \prec E^{<1>} + E^{<h>} \succ E^{<1>}) \\
&= E^{<g>} \prec (E^{<h>} \prec E^{<1>}) + E^{<g>} \prec (E^{<h>} \succ E^{<1>}) + E^{<g>} \succ (E^{<h>} \prec E^{<1>}) \\
&\quad + E^{<g>} \succ (E^{<h>} \succ E^{<1>}) = (E^{<g>} \prec E^{<h>}) \prec E^{<1>} + (E^{<g>} \succ E^{<h>}) \prec E^{<1>} \\
&\quad + ((E^{<g>} \prec E^{<h>}) \succ E^{<1>}) + (E^{<g>} \succ E^{<h>}) \succ E^{<1>} = (E^{<g>} \lozenge E^{<h>}) \prec E^{<1>} \\
&\quad + (E^{<g>} \lozenge E^{<h>}) \succ E^{<1>} = (E^{<g>} \lozenge E^{<h>}) \lozenge E^{<1>} \subseteq E^{<g+h>} \lozenge E^{<1>} \subseteq E^{<g+h+1>}.
\end{aligned}$$

The proof of (b) is similar. □

Lemma 2.3. *Let E be a dendriform algebra. Then for any* $g \in \mathbb{N}$, *we have*:

$$E^{<g>} = E^{\{g\}} = E^g.$$

Proof. We show that $E^{<g>} = E^g$, other case can be shown by similar way. From (2.2) and (2.4), obviously that $E^{<g>} \subseteq E^g$. From (2.5), we know that $E^{<g>} \Diamond E^{<h>} \subseteq E^{<g+h>}$. It is not difficult to see that $E^{<1>} = E^1$ and $E^{<2>} = E^2$. Using mathematical induction method for $g \in \mathbb{N}$, we can assume that $E^{<g>} = E^g$, and we have to proof for $g+1$. Formula (2.4) give us the following: $E^{g+1} = E^1 \Diamond E^g + E^2 \Diamond E^{g-1} + \ldots + E^g \Diamond E^1 \subseteq E^{<1>} \Diamond E^{<g>} + E^{<2>} \Diamond E^{<g-1>} + \ldots + E^{<g>} \Diamond E^{<1>} \subseteq E^{<g+1>}$. □

From Lemma 3.3, it follows that, in dendriform algebras notions of the left nilpotency, the right nilpotency and the nilpotency coincide.

Proposition 2.1. *Dendriform algebra* $(E, \prec, \succ)$ *is nilpotent, if and only if* $(E, \prec)$ *and* $(E, \succ)$ *are nilpotent.*

Proof. It is clear that from the nilpotency of $(E, \prec, \succ)$, follows the nilpotency of $(E, \prec)$ and $(E, \succ)$.

Now, let $(E, \prec)$ and $(E, \succ)$ be nilpotent algebras, i.e., there exist $s, t \in \mathbb{N}$ such that $E_{\prec}^s = 0$ and $E_{\succ}^t = 0$. Then, for $p \geq s+t-2$, we have $E^p = 0$. Indeed, an element $x \in E^p$ can be written in the form:

$$\begin{aligned} x = {} & (((x_1 \Diamond x_2) \Diamond x_3) \Diamond \ldots \Diamond x_{p-1}) \Diamond x_p = (((x_1 \prec x_2) \prec x_3) \prec \ldots \prec x_{p-1}) \prec x_p \\ & + (((x_1 \succ x_2) \prec x_3) \prec \ldots \prec x_{p-1}) \prec x_p + \ldots + (((x_1 \succ x_2) \succ x_3) \succ \ldots \prec x_{p-1}) \prec x_p \\ & + (((x_1 \succ x_2) \succ x_3) \succ \ldots \succ x_{p-1}) \prec x_p + (((x_1 \succ x_2) \succ x_3) \succ \ldots \succ x_{p-1}) \succ x_p. \end{aligned} \tag{2.7}$$

It is easy to see that, if in each term of the summands there are at least s time $\prec$ or t times $\succ$, then it is zero, since s and t are nilindexes of $(E, \prec)$ and $(E, \succ)$, respectively. Otherwise, times of $\succ$ and $\prec$ can be increased by using the axioms:

$$x_l \prec (x_m \succ x_n) = (x_l \prec x_m) \prec x_n - x_l \prec (x_m \prec x_n), \tag{2.8}$$

$$(x_l \prec x_m) \succ x_n = x_l \succ (x_m \succ x_n) - (x_l \succ x_m) \succ x_n. \tag{2.9}$$

$\square$

3. Isomorphism Classes of Two Dimensional Dendriform Algebras

In this section, we discover all dendriform algebra structures on 2-dimensional complex vector space. Here, we keep the notation $Dend_n^q$ for q-th isomorphism class of dendriform algebra structures in n-dimensional vector space.

Theorem 3.1. *Any two-dimensional dendriform algebra can be included in one of the following classes of algebras*:

$Dend_2^1(\alpha): e_1 \prec e_1 = e_2,\ e_1 \succ e_1 = \alpha e_2,\ \alpha \in \mathbb{C};$

$Dend_2^2: e_1 \prec e_1 = e_1,\ e_1 \succ e_2 = e_2;$

$Dend_2^3: e_1 \prec e_1 = e_1,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_2 = e_2,\ e_2 \succ e_1 = -e_2;$

$Dend_2^4: e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1;$

$Dend_2^5: e_1 \prec e_2 = -e_2,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1,\ e_1 \succ e_2 = e_2;$

$Dend_2^6: e_1 \prec e_1 = e_1,\ e_2 \succ e_2 = e_2;$

$Dend_2^7: e_1 \prec e_2 = -e_2,\ e_2 \prec e_2 = e_2,\ e_1 \succ e_1 = e_1,\ e_1 \succ e_2 = e_2;$

$Dend_2^8: e_1 \prec e_1 = e_1 + e_2,\ e_1 \prec e_2 = -e_2,\ e_2 \prec e_2 = e_2,\ e_1 \succ e_1 = -e_2,$
$e_1 \succ e_2 = e_2;$

$Dend_2^9: e_1 \prec e_1 = e_1,\ e_2 \prec e_1 = e_2,\ e_2 \succ e_1 = -e_2,\ e_2 \succ e_2 = e_2;$

$Dend_2^{10}: e_1 \prec e_1 = -e_2,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1 + e_2,\ e_2 \succ e_1 = -e_2,$
$e_2 \succ e_2 = e_2;$

$Dend_2^{11}: e_2 \prec e_1 = e_2, e_1 \succ e_1 = e_1, e_1 \succ e_2 = e_2;$

$Dend_2^{12}: e_1 \prec e_1 = e_1, e_2 \prec e_1 = e_2, e_1 \succ e_2 = e_2.$

Proof. Let E be a two dimensional dendriform algebra. Then from (2.1), we have

$$e_1 \prec e_1 = \alpha_{11}^1 e_1 + \alpha_{11}^2 e_2, \quad e_1 \prec e_2 = \alpha_{12}^1 e_1 + \alpha_{12}^2 e_2,$$

$$e_2 \prec e_1 = \alpha_{21}^1 e_1 + \alpha_{21}^2 e_2, \quad e_2 \prec e_2 = \alpha_{22}^1 e_1 + \alpha_{22}^2 e_2,$$

$$e_1 \succ e_1 = \beta_{11}^1 e_1 + \beta_{11}^2 e_2, \quad e_1 \succ e_2 = \beta_{12}^1 e_1 + \beta_{12}^2 e_2,$$

$$e_2 \succ e_1 = \beta_{21}^1 e_1 + \beta_{21}^2 e_2, \quad e_2 \succ e_2 = \beta_{22}^1 e_1 + \beta_{22}^2 e_2.$$

Let us introduce the following notations:

$$\alpha_{11}^1 = \alpha_1, \alpha_{11}^2 = \alpha_2, \alpha_{12}^1 = \alpha_3, \alpha_{12}^2 = \alpha_4, \alpha_{21}^1 = \alpha_5, \alpha_{21}^2 = \alpha_6, \alpha_{22}^1 = \alpha_7,$$
$$\alpha_{22}^2 = \alpha_8, \ \beta_{11}^1 = \beta_1, \beta_{11}^2 = \beta_2, \beta_{12}^1 = \beta_3, \beta_{12}^2 = \beta_4, \beta_{21}^1 = \beta_5, \beta_{21}^2 = \beta_6,$$
$$\beta_{22}^1 = \beta_7, \beta_{22}^2 = \beta_8. \tag{3.1}$$

Verifying the dendriform algebra identities, we get the following system of equations for the structure constants α_i, β_i for $1 \le i \le 8$.

1. $\alpha_5\beta_2 = \alpha_2\beta_3$, 2. $\alpha_4\beta_7 = \alpha_7\beta_6$, 3. $\alpha_7\beta_2 = \alpha_4\beta_3$, 4. $\alpha_5\beta_6 = \alpha_2\beta_7$,

5. $\alpha_2\alpha_6 = \alpha_2\alpha_4 + \alpha_2\beta_1 + \alpha_4\beta_2$, 6. $\alpha_3\alpha_7 = \alpha_5\alpha_7 + \alpha_5\beta_7 + \alpha_7\beta_8$,

7. $\alpha_1\beta_3 + \alpha_5\beta_4 = \alpha_5\beta_1 + \alpha_6\beta_3$, 8. $\alpha_1\beta_7 + \alpha_5\beta_8 = \alpha_5\beta_5 + \alpha_6\beta_7$,

9. $\alpha_2\beta_1 + \alpha_6\beta_2 = \alpha_1\beta_2 + \alpha_2\beta_4$, 10. $\alpha_2\beta_5 + \alpha_6\beta_6 = \alpha_1\beta_6 + \alpha_2\beta_8$,

11. $\alpha_3\beta_7 + \alpha_7\beta_8 = \alpha_7\beta_5 + \alpha_8\beta_7$, 12. $\alpha_3\beta_3 + \alpha_7\beta_4 = \alpha_7\beta_1 + \alpha_8\beta_3$,

13. $\alpha_4\beta_1 + \alpha_8\beta_2 = \alpha_3\beta_2 + \alpha_4\beta_4$, 14. $\alpha_4\beta_5 + \alpha_8\beta_6 = \alpha_3\beta_6 + \alpha_4\beta_8$,

15. $\alpha_7\beta_3 + \alpha_8\beta_7 + \beta_3\beta_7 = \beta_5\beta_7$, 16. $\alpha_1\beta_2 + \alpha_2\beta_6 + \beta_2\beta_6 = \beta_2\beta_4$,

17. $\alpha_1\beta_1 + \alpha_2\beta_5 + \alpha_3\beta_2 + \alpha_4\beta_6 = 0$, 18. $\alpha_1\beta_3 + \alpha_2\beta_7 + \alpha_3\beta_3 + \alpha_4\beta_8 = 0$,

19. $\alpha_5\beta_1 + \alpha_6\beta_5 + \alpha_7\beta_2 + \alpha_8\beta_6 = 0$, 20. $\alpha_5\beta_3 + \alpha_6\beta_7 + \alpha_7\beta_4 + \alpha_8\beta_8 = 0$,

21. $\alpha_1\alpha_3 + \alpha_4\alpha_5 = \alpha_1\alpha_5 + \alpha_3\alpha_6 + \alpha_1\beta_5 + \alpha_3\beta_6$,

22. $\alpha_1\alpha_4 + \alpha_2\alpha_8 = \alpha_2\alpha_3 + \alpha_4^2 + \alpha_2\beta_3 + \alpha_4\beta_4$,

23. $\alpha_1\alpha_7 + \alpha_5\alpha_8 = \alpha_5^2 + \alpha_6\alpha_7 + \alpha_5\beta_5 + \alpha_7\beta_6$,

24. $\alpha_2\alpha_5 + \alpha_6^2 = \alpha_1\alpha_6 + \alpha_2\alpha_8 + \alpha_6\beta_1 + \alpha_8\beta_2$,

25. $\alpha_3^2 + \alpha_4\alpha_7 = \alpha_1\alpha_7 + \alpha_3\alpha_8 + \alpha_1\beta_7 + \alpha_3\beta_8$,

26. $\alpha_4\alpha_5 + \alpha_6\alpha_8 = \alpha_3\alpha_6 + \alpha_4\alpha_8 + \alpha_6\beta_3 + \alpha_8\beta_4$,

27. $\alpha_1\beta_4 + \alpha_2\beta_8 + \beta_1\beta_4 + \beta_2\beta_8 = \beta_2\beta_3 + \beta_4^2$,

28. $\alpha_3\beta_1 + \alpha_4\beta_5 + \beta_1\beta_3 + \beta_4\beta_5 = \beta_1\beta_5 + \beta_3\beta_6$,

29. $\alpha_3\beta_3 + \alpha_4\beta_7 + \beta_3^2 + \beta_4\beta_7 = \beta_1\beta_7 + \beta_3\beta_8$,

30. $\alpha_5\beta_2 + \alpha_6\beta_6 + \beta_2\beta_5 + \beta_6^2 = \beta_1\beta_6 + \beta_2\beta_8$,

31. $\alpha_5\beta_4 + \alpha_6\beta_8 + \beta_4\beta_5 + \beta_6\beta_8 = \beta_3\beta_6 + \beta_4\beta_8$,

32. $\alpha_7\beta_1 + \alpha_8\beta_5 + \beta_1\beta_7 + \beta_5\beta_8 = \beta_5^2 + \beta_6\beta_7$.

From Lemma 2.1 and formulas (1.2), (2.1), and (3.1), we have

$$e_i * e_j = e_i \prec e_j + e_i \succ e_j,\ \gamma_i = \alpha_i + \beta_i. \tag{3.2}$$

Let us consider the associative algebra As_2^1 from Theorem 1.1, with the table of multiplication:

$$e_1 * e_1 = e_2.$$

Then, it is not difficult to see that $\gamma_2 = 1$ and the other structure constants are equal to zero. So from (3.2), we obtain

$$\alpha_1 = -\beta_1,\ \beta_2 = 1 - \alpha_2,\ \alpha_3 = -\beta_3,\ \alpha_4 = -\beta_4,$$

$$\alpha_5 = -\beta_5,\ \alpha_6 = -\beta_6,\ \alpha_7 = -\beta_7,\ \alpha_8 = -\beta_8.$$

Substituting to the above Equations 1-32, we get the following conditions for structure constants α_i, for $1 \le i \le 8$.

$$\alpha_3^2 = \alpha_5^2,\ \alpha_2\alpha_7 = \alpha_5\alpha_6,\ \alpha_3\alpha_7 = \alpha_5\alpha_7,\ \alpha_3\alpha_8 = \alpha_5\alpha_8,\ \alpha_4\alpha_7 = \alpha_6\alpha_7,$$

$$\alpha_4\alpha_7 + \alpha_3^2 = 0,\ \alpha_1\alpha_3 + \alpha_4\alpha_5 = 0,\ \alpha_1\alpha_4 + \alpha_2\alpha_8 = 0,\ \alpha_1\alpha_5 + \alpha_3\alpha_6 = 0,$$

$$\alpha_1\alpha_7 + \alpha_3\alpha_8 = 0, \alpha_3\alpha_6 + \alpha_4\alpha_8 = 0,\ \alpha_3\alpha_7 + \alpha_7\alpha_8 = 0,\ \alpha_4\alpha_5 + \alpha_6\alpha_8 = 0,$$

$$\alpha_2\alpha_5 + \alpha_6^2 = \alpha_8,\ \alpha_5 + \alpha_2\alpha_3 = \alpha_2\alpha_5,\ \alpha_7 + \alpha_3\alpha_4 = \alpha_2\alpha_7,$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_6 = \alpha_4,\ \alpha_1\alpha_6 + \alpha_2\alpha_8 = \alpha_8,\ \alpha_1\alpha_5 + \alpha_6\alpha_8 = \alpha_7,$$

$$\alpha_8 + \alpha_2\alpha_3 + \alpha_4^2 = \alpha_3,\ \alpha_3\alpha_5 + 2\alpha_4\alpha_7 + \alpha_8^2 = 0,\ \alpha_4^2 + \alpha_6^2 = \alpha_1^2 + \alpha_4\alpha_6,$$

$$\alpha_1 + \alpha_2\alpha_6 = \alpha_6 + \alpha_2\alpha_4,\ \alpha_1\alpha_3 + \alpha_2\alpha_7 + \alpha_3^2 + \alpha_4\alpha_8 = 0,$$

$$\alpha_1\alpha_5 + \alpha_2\alpha_7 + \alpha_3\alpha_4 + \alpha_6\alpha_8 = 0.$$

There are two possible cases for $\alpha_3 : \alpha_3 = \alpha_5$, and $\alpha_3 = -\alpha_5$. An easy calculation shows that in the both cases, we have

$$\alpha_1 = 0,\ \alpha_3 = 0,\ \alpha_4 = 0,\ \alpha_5 = 0,\ \alpha_6 = 0,\ \alpha_7 = 0,\ \alpha_8 = 0,$$

$$\beta_1 = 0,\ \beta_3 = 0,\ \beta_4 = 0,\ \beta_5 = 0,\ \beta_6 = 0,\ \beta_7 = 0,\ \beta_8 = 0,\ \beta_2 = 1 - \alpha_2.$$

Taking α for α_2, we get the following continues family of dendriform algebras:

$$e_1 \prec e_1 = \alpha e_2, e_1 \succ e_1 = (1-\alpha)e_2.$$

It can be noticed that the algebras for different values of parameter α are never isomorphic.

The base change $e_1' = e_1$, $e_2' = \alpha e_2 (\alpha \neq 0)$, with $\frac{1-\alpha}{\alpha} \leftrightarrow \alpha$, leads to

$$Dend_2^1 : e_1 \prec e_1 = e_2,\ e_1 \succ e_1 = \alpha e_2,\ \alpha \in \mathbb{C}.$$

Let us now consider the algebra $As_2^2 : e_1 * e_1 = e_1, e_1e_2 = e_2$ from the list of Theorem 1.1. Then, the conditions (3.2) give the constraints for the structure constants as follows:

$$\beta_1 = 1 - \alpha_1, \beta_2 = -\alpha_2, \beta_3 = -\alpha_3, \beta_4 = 1 - \alpha_4,$$

$$\beta_5 = -\alpha_5, \beta_6 = -\alpha_6, \beta_7 = -\alpha_7, \beta_8 = -\alpha_8. \tag{3.3}$$

After substitution value of β_i, for $1 \le i \le 8$ to the system of Equations 1-32, we have:

$$\alpha_3^2 = \alpha_5^2, \alpha_2\alpha_3 = \alpha_2\alpha_5, \alpha_2\alpha_7 = \alpha_3\alpha_4, \alpha_2\alpha_7 = \alpha_5\alpha_6, \alpha_2\alpha_4 = \alpha_2\alpha_6,$$

$$\alpha_3\alpha_7 = \alpha_5\alpha_7, \alpha_3\alpha_8 = \alpha_5\alpha_8, \alpha_4\alpha_7 = \alpha_6\alpha_7, \alpha_3^2 + \alpha_4\alpha_7 = 0, \alpha_1\alpha_3 + \alpha_4\alpha_5 = 0,$$

$$\alpha_1\alpha_5 + \alpha_3\alpha_6 = 0, \alpha_1\alpha_7 + \alpha_3\alpha_8 = 0, \alpha_3\alpha_7 + \alpha_7\alpha_8 = 0, \alpha_5\alpha_7 + \alpha_7\alpha_8 = 0,$$

$$\alpha_2\alpha_3 + \alpha_4^2 = \alpha_4, \alpha_2\alpha_5 + \alpha_6^2 = \alpha_6, \alpha_1\alpha_2 + \alpha_2\alpha_4 = \alpha_2, \alpha_1\alpha_4 + \alpha_2\alpha_8 = \alpha_4,$$

$$\alpha_1\alpha_6 + \alpha_2\alpha_8 = \alpha_6, \alpha_3\alpha_6 + \alpha_4\alpha_8 = \alpha_8, \alpha_4\alpha_5 + \alpha_6\alpha_8 = \alpha_8,$$

$$\alpha_1^2 + 2\alpha_2\alpha_3 + \alpha_4\alpha_6 = \alpha_1, \alpha_3\alpha_5 + 2\alpha_4\alpha_7 + \alpha_8^2 = \alpha_7,$$

$$\alpha_1\alpha_5 + 2\alpha_2\alpha_7 + \alpha_6\alpha_8 = \alpha_5, \alpha_1\alpha_3 + \alpha_2\alpha_7 + \alpha_3^2 + \alpha_4\alpha_8 = 0.$$

Several cases and subcases may occur here.

Case 1. Let $\alpha_3 = \alpha_5$, then $\alpha_3 = 0, \alpha_5 = 0, \alpha_7 = 0, \alpha_8 = 0,$ and

$$\alpha_4^2 = \alpha_4, \alpha_6^2 = \alpha_6, \alpha_1\alpha_4 = \alpha_4, \alpha_1\alpha_6 = \alpha_6,$$

$$\alpha_2\alpha_4 = \alpha_2\alpha_6, \alpha_1^2 + \alpha_4\alpha_6 = \alpha_1, \alpha_1\alpha_2 + \alpha_2\alpha_4 = \alpha_2.$$

Case 1.1. If $\alpha_4 = 0,$ then we have:

$$\alpha_1^2 = \alpha_1, \alpha_6^2 = \alpha_6, \alpha_2\alpha_6 = 0, \alpha_1\alpha_6 = \alpha_6, \text{ and } \alpha_1\alpha_2 = \alpha_2.$$

Case 1.1.1. Assume that $\alpha_1 = 0,$ then we get $\alpha_2 = 0, \alpha_6 = 0.$ In this case, we get an associative algebra.

Case 1.1.2. Let us suppose that $\alpha_1 \neq 0$, then $\alpha_1 = 1$ and we get $\alpha_6^2 = \alpha_6$, $\alpha_2\alpha_6 = 0$.

Case 1.1.2.1. If $\alpha_6 = 0$, then we get the following table of multiplication:

$$e_1 \prec e_1 = e_1 - \alpha_2 e_2,\ e_1 \succ e_1 = \alpha_2 e_2,\ e_1 \succ e_2 = e_2.$$

The base change $e_1' = e_1 - \alpha_2 e_2$, $e_2' = e_2$ leads to the algebra:

$$Dend_2^2 : e_1 \prec e_1 = e_1,\ e_1 \succ e_2 = e_2.$$

Case 1.1.2.2. Let $\alpha_6 \neq 0$, then $\alpha_6 = 1$ and $\alpha_2 = 0$. Hence, one has the following algebra:

$$Dend_2^3 : e_1 \prec e_1 = e_1,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_2 = e_2,\ e_2 \succ e_1 = -e_2.$$

Case 1.2. Let say $\alpha_4 \neq 0$, then $\alpha_4 = 1$, $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_6 = 0$. The resulting algebra is associative.

Case 2. Consider the case when $\alpha_3 = -\alpha_5$, then $\alpha_3 = 0$, $\alpha_5 = 0$, $\alpha_7 = 0$, and $\alpha_8 = 0$, we obtain the similar constraints for the structure constants as those in **Case 1**.

Let us now consider the algebra $As_3^2 : e_1 * e_1 = e_1$ and $e_2 * e_1 = e_2$ from the list of Theorem 1.1. Then, applying (3.2) one can easy to get the following conditions for structure constants:

$$\beta_1 = 1 - \alpha_1,\ \beta_2 = -\alpha_2,\ \beta_3 = -\alpha_3,\ \beta_4 = -\alpha_4,$$

$$\beta_5 = -\alpha_5,\ \beta_6 = 1 - \alpha_6,\ \beta_7 = -\alpha_7,\ \beta_8 = -\alpha_8.$$

Substituting again, after some algebra, we get the system of equation as follows:

$$\alpha_3 = 0,\ \alpha_5 = 0,\ \alpha_7 = 0,\ \alpha_8 = 0,$$

$$\alpha_6 = \alpha_6^2,\ \alpha_1\alpha_4 = 0,\ \alpha_1\alpha_6 = \alpha_6,\ \alpha_4^2 + \alpha_4 = 0,$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_4 = 0,\ \alpha_1\alpha_2 + \alpha_2\alpha_6 = \alpha_2,\ \alpha_1^2 + \alpha_4\alpha_6 = \alpha_1 + \alpha_4.$$

Case 3. The case $\alpha_6 = 0$ gives $\alpha_1 = 0, \alpha_2 = 0, \alpha_4 = 0,$ and the resulting algebra again is associative.

Case 4. Let $\alpha_6 \neq 0$. Then $\alpha_6 = 1$ and

$$\alpha_1 = \alpha_1^2,\ \alpha_1\alpha_2 = 0,\ \alpha_1\alpha_4 = 0,\ \alpha_2\alpha_4 = 0, \text{ and } \alpha_4^2 + \alpha_4 = 0.$$

Case 4.1. If $\alpha_1 = 0$, then

$$\alpha_2\alpha_4 = 0,\ \alpha_4^2 + \alpha_4 = 0.$$

Case 4.1.1. If $\alpha_4 = 0$, then α_2 is free, and taking it as a parameter α, we get the multiplication table:

$$e_1 \prec e_1 = \alpha e_2,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1 - \alpha e_2.$$

Changing the basis $\{e_1, e_2\}$ as $e_1' = e_1 - \alpha e_2, e_2' = e_2$, the above table can be reduced to:

$$Dend_2^4 : e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1.$$

Case 4.1.2. Let $\alpha_4 \neq 0$. Then $\alpha_2 = 0$ and $\alpha_4 = -1$. We obtain the algebra:

$$Dend_2^5 : e_1 \prec e_2 = -e_2,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1,\ e_1 \succ e_2 = e_2.$$

Case 4.2. If $\alpha_1 \neq 0$, then $\alpha_1 = 1, \alpha_2 = 0,$ and $\alpha_4 = 0$. It is not difficult to see that, in this case, the algebra is associative.

Consider the algebra $As_4^2 : e_1 * e_1 = e_1, e_2 * e_2 = e_2$. Then, the conditions for structure constants be as follows:

$$\beta_1 = 1 - \alpha_1,\ \beta_2 = -\alpha_2,\ \beta_3 = -\alpha_3,\ \beta_4 = -\alpha_4,$$

$$\beta_5 = -\alpha_5,\ \beta_6 = -\alpha_6,\ \beta_7 = -\alpha_7,\ \beta_8 = 1 - \alpha_8.$$

As in the above considered cases, we get the following system of equations:

$$\alpha_3 = 0,\ \alpha_5 = 0,\ \alpha_7 = 0,$$

$$\alpha_6 = \alpha_6^2,\ \alpha_8 = \alpha_8^2,\ \alpha_6\alpha_8 = 0,\ \alpha_4\alpha_8 = \alpha_4,\ \alpha_4^2 + \alpha_4 = 0,$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_4 = 0,\ \alpha_1\alpha_4 + \alpha_2\alpha_8 = 0,\ \alpha_1^2 + \alpha_4\alpha_6 = \alpha_1,$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_6 = \alpha_2,\ \alpha_2\alpha_8 + \alpha_1\alpha_6 = \alpha_2 + \alpha_6.$$

Case 5. If $\alpha_6 = 0$, then

$$\alpha_1 = \alpha_1^2,\ \alpha_8 = \alpha_8^2,\ \alpha_1\alpha_2 = \alpha_2,\ \alpha_4\alpha_8 = \alpha_4,\ \alpha_4^2 + \alpha_4 = 0,$$

$$\alpha_2 + \alpha_8 = \alpha_2,\ \alpha_1\alpha_2 + \alpha_2\alpha_4 = 0,\ \alpha_1\alpha_4 + \alpha_2\alpha_8 = 0.$$

Case 5.1. Let $\alpha_8 = 0$, then $\alpha_2 = 0, \alpha_4 = 0$, and $\alpha_1 = \alpha_1^2$. In this case, $\alpha_1 = 1$ (since otherwise, at $\alpha_1 = 0$, the algebra is associative) therefore, one gets the following table of multiplication:

$$Dend_2^6 : e_1 \prec e_1 = e_1,\ e_2 \succ e_2 = e_2.$$

Case 5.2. Let now $\alpha_8 \neq 0$. Then $\alpha_8 = 1$ and this implies

$$\alpha_1 = \alpha_1^2,\ \alpha_1\alpha_2 = \alpha_2,\ \alpha_4^2 + \alpha_4 = 0,$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_4 = 0,\ \alpha_1\alpha_4 + \alpha_2 = 0.$$

Case 5.2.1. Let $\alpha_1 = 0$, then $\alpha_2 = 0$ and

$$\alpha_4^2 + \alpha_4 = 0.$$

Here, if $\alpha_4 = 0$, then we get the table:

$$e_2 \prec e_2 = e_2,\ e_1 \succ e_1 = e_1.$$

It is easy to see that the base change $e_1' = e_2, e_2' = e_1$ leads to $Dend_2^6$.

If $\alpha_4 \neq 0$, then $\alpha_4 = -1$ and we have the algebra:

$$Dend_2^7 : e_1 \prec e_2 = -e_2,\ e_2 \prec e_2 = e_2,\ e_1 \succ e_1 = e_1,\ e_1 \succ e_2 = e_2.$$

Case 5.2.2. If $\alpha_1 \neq 0$, then $\alpha_1 = 1$ and

$$\alpha_4^2 + \alpha_4 = 0,\ \alpha_2 + \alpha_2\alpha_4 = 0,\ \alpha_2 + \alpha_4 = 0.$$

In this case, if $\alpha_4 = 0$, then $\alpha_2 = 0$ and the algebra is associative.

If $\alpha_4 \neq 0$, then $\alpha_4 = -1$, $\alpha_2 = 1$, and this is the case of the algebra:

$$Dend_2^8 : e_1 \prec e_1 = e_1 + e_2,\ e_1 \prec e_2 = -e_2,\ e_2 \prec e_2 = e_2,\ e_1 \succ e_1 = -e_2,$$

$$e_1 \succ e_2 = e_2.$$

Case 6. Let $\alpha_6 \neq 0$. Then $\alpha_6 = 1$, $\alpha_4 = 0$, $\alpha_8 = 0$, and

$$\alpha_1 = \alpha_1^2,\ \alpha_1\alpha_2 = 0,\ \alpha_1 = \alpha_2 + 1.$$

Case 6.1. Let $\alpha_2 = 0$, then $\alpha_1 = 1$, in this case, we obtain the following algebra:

$$Dend_2^9 : e_1 \prec e_1 = e_1,\ e_2 \prec e_1 = e_2,\ e_2 \succ e_1 = -e_2,\ e_2 \succ e_2 = e_2.$$

Case 6.2. Let $\alpha_2 \neq 0$, then $\alpha_1 = 0$, $\alpha_2 = -1$, so we have the algebra:

$$Dend_2^{10} : e_1 \prec e_1 = -e_2,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1 + e_2,\ e_2 \succ e_1 = -e_2,$$

$$e_2 \succ e_2 = e_2.$$

Now, let us consider algebra $As_5^2 : e_1 * e_1 = e_1$, $e_1 * e_2 = e_2$, and $e_2 * e_1 = e_2$. Then, we obtain the conditions for structure constants as follows:

$$\alpha_3 = 0,\ \alpha_5 = 0,\ \alpha_7 = 0,\ \alpha_8 = 0,$$

$$\alpha_4 = \alpha_4^2,\ \alpha_6 = \alpha_6^2,\ \alpha_1\alpha_4 = \alpha_4,\ \alpha_1\alpha_6 = \alpha_1,$$

$$\alpha_2\alpha_4 = \alpha_2\alpha_6,\ \alpha_1\alpha_2 + \alpha_2\alpha_4 = \alpha_2,\ \alpha_1 + \alpha_4 = \alpha_4\alpha_6 + \alpha_1^2.$$

The following two $\alpha_4 = 0$ and $\alpha_4 = 1$ cases are occurred here. If $\alpha_4 = 0$, then we get the dendriform algebras with tables:

$$Dend_2^{11} : e_2 \prec e_1 = e_2,\ e_1 \succ e_1 = e_1,\ e_1 \succ e_2 = e_2,$$

$$Dend_2^{12} : e_1 \prec e_1 = e_1,\ e_2 \prec e_1 = e_2,\ e_1 \succ e_2 = e_2.$$

In the case of $\alpha_4 = 1$, we obtain an associative algebra.

There is a result of Omirov [7] on classification of 2-dimensional complex Zenbiel algebras. According to this result, there exists only one two-dimensional non-trivial complex Zinbiel algebra, with the following table of multiplication:

$$Zinb_2^1 : e_1e_1 = e_2.$$

This algebra can be derived from $Dend_2^1$ at $\alpha = 1$. □

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